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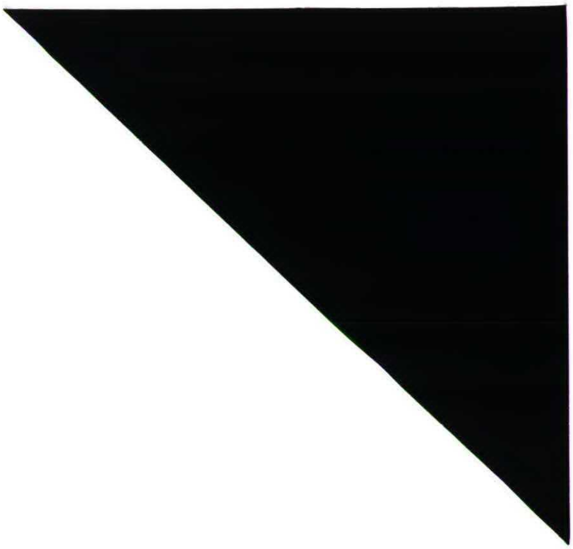

# AN ASYMPTOTIC JUSTIFICATION FOR A MODIFIED GLS PROCEDURE TO ESTIMATE ARMA PARAMETERS

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# AN ASYMPTOTIC JUSTIFICATION FOR A MODIFIED GLS PROCEDURE TO ESTIMATE ARMA PARAMETERS

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**Abstract**

The asymptotic form of the concentrated likelihood function of a stationary ARMA model is investigated. Basis for the analysis is the closed form formula for the exact covariance matrix. GLS estimates of regression coefficients can be based on a simple and widely used adjusted form of the covariance matrix. However, ML-estimates of the ARMA parameters become more complicated, as they depend on the limit of the determinant of the covariance matrix.

JEL code: C22

Keywords : Generalised Linear Model; Autoregressive moving average process; exact ARMA covariance matrix.

## 1. Introduction

Several authors have presented procedures to estimate the covariance parameters, be it for a pure time series model (Kohn and Ansley, 1985) or for a regression model (Zinde-Walsh and Galbraith, 1991). The resulting formulas and algorithms become very complicated. In Van der Leeuw (1993) relatively simple expressions are presented, based on a closed form of the general ARMA covariance matrix. This is possible as the closed form of the covariance matrix is simple enough to be differentiated analytically. The price to be paid for this procedure is the computation of the covariance matrix and its inverse. Moreover, second derivatives need very much (computer) time consuming algorithms, as they depend on the complete covariance matrix.

In this paper the asymptotic form of the covariance matrix and its determinant are studied. Maximum likelihood estimation of the ARMA model is equivalent to determining the first order condition of the concentrated likelihood. In this expression the covariance matrix appears twice: as determinant and as part of a quadratic form. We will show what happens with both forms when the number of observations becomes large.

## 2. The linear model

Consider the linear model $y=X\beta+\varepsilon$, where y has dimensions $(T\times 1)$, X $(T\times k)$, $\beta$ $(k\times 1)$ and $\varepsilon$ $(T\times 1)$ and covariance matrix $E\varepsilon\varepsilon^T=\sigma^2 V$, which is a function of a vector $\vartheta$. One way to estimate the unknown parameters $\beta$ and $\vartheta$ is by minimizing the weighted sum of squares $\varepsilon^T V^{-1}\varepsilon$. Supposing normally distributed errors, we may prefer to maximize the (concentrated) likelihood function, which is equivalent to minimizing $S=|V|^{1/T}e^T V^{-1}e$ (Judge *et al.*, p.284). Here $e=y-Xb$, where b is the Aitken estimator of $\beta$.

Minimizing $S=|V|^{1/T}e^{T}V^{-1}e$ is equivalent to solving the first order conditions or to find a solution to

$$\mathbf{d}S=|V|^{1/T}\{s^2\mathbf{tr}V^{-1}\mathbf{d}V+e^{T}\mathbf{d}(V^{-1})e\}=0 \tag{1}$$

with $s^2=e^{T}V^{-1}e/T$. In the next section we show how V can be written in a form that can be differentiated easily and substituted into (1). The pure time series model is similar: e need not to be computed as it is identical to y.

### 3. The ARMA covariance matrix

The general form of ARMA distributed errors is given by

$$\varepsilon_t=-\sum_{i=1}^{p}\vartheta_i\varepsilon_{t-i}+v_t+\sum_{i=1}^{q}\alpha_i v_{t-i},\quad t=1,..,T, \tag{2}$$

where $v_t$ is a sequence of independently and identically distributed random variables. $\vartheta$ denotes the vector $(\vartheta_1,\vartheta_2,..,\vartheta_p)^{T}$ of AR-parameters, $\alpha$ is the vector $(\alpha_1,\alpha_2,..,\alpha_q)^{T}$ of MA-parameters. We assume that p and q are known and that the invertibility conditions are fulfilled. By definition $\vartheta_0$ and $\alpha_0$ are put equal to 1. Use $\sigma^2V$ to denote the covariance matrix of $\varepsilon$: $\sigma^2V=E\varepsilon\varepsilon^{T}$.

Following Pagan (1974), we introduce two matrices for both the AR parameters and the MA parameters. We define a (square) lower band matrix P of dimensions TxT, and a Txp matrix Q as follows:

$$P=\begin{bmatrix}1 & & & & & \\ \vartheta_1 & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & & & & & \\ \vartheta_p & & & & & \\ & \cdot & & \cdot & \cdot & \\ & & \vartheta_p\cdot & & \cdot\ \vartheta_1 & 1\end{bmatrix},\quad Q=\begin{bmatrix}\vartheta_p & \vartheta_{p-1} & \cdot & \vartheta_1\\ 0 & \cdot & \cdot & \cdot\\ \cdot & \cdot & & \\ & & & \cdot\\ 0 & & & \vartheta_p\\ 0 & \cdot & \cdot & 0\\ \cdot & & & \cdot\\ 0 & \cdot & \cdot & 0\end{bmatrix}.$$

The upper triangular part of a lower band matrix consists of zeros and the lower part has off-diagonals with the same elements. Q consists of an upper p×p part with an upper band matrix and a lower (T−p)×p part, which consists of only zeros. Like P and Q will be used to describe the AR part of the error vector, so are M and N defined for the MA part, where $\vartheta$ is replaced by $\alpha$ and p by q.

Defining the auxiliary vectors $\bar{\varepsilon}=(\varepsilon_{-p+1},\varepsilon_{-p+2},\dots,\varepsilon_{-1},\varepsilon_{0})^{T}$ and $\bar{v}=(v_{-q+1},v_{-q+2},\dots,v_{-1},v_{0})^{T}$, we can write (2) in matrix form:

$$\begin{bmatrix} Q & P \end{bmatrix}\begin{bmatrix} \bar{\varepsilon} \\ \varepsilon \end{bmatrix}=\begin{bmatrix} N & M \end{bmatrix}\begin{bmatrix} \bar{v} \\ v \end{bmatrix}.$$

As is proven elsewhere (Van der Leeuw, 1992) in the exact covariance matrix, $E\varepsilon\varepsilon^{T}=\sigma^{2}V$, of ARMA errors, V is equal to $[N\ M][\bar{P}^{T}\bar{P}-\bar{Q}\bar{Q}^{T}]^{-1}[N\ M]^{T}$, where $\bar{P}$ is like P, but of order (T+p)×(T+p) and $\bar{Q}$ like Q, but of order (T+p)×p. In the MA case this expression reduces to $[N\ M][N\ M]^{T}$ and in the AR case it becomes $[P^{T}P-QQ^{T}]^{-1}$. When we use the underlined symbols $\underline{P}$, $\underline{Q}$, $\underline{M}$, and $\underline{N}$, we mean the upper left (p×p) submatrix of corresponding matrix. Furthermore we will assume p greater than or equal to q. This gives no loss of generality as the case q≥p is similar.

**4. Limits of quadratic form and determinant**

As stated before the first order conditions for the AR and MA parameters are derived from the differential $\mathbf{d}S=|V|^{1/T}\{s^{2}\mathbf{tr}V^{-1}\mathbf{d}V+e^{T}\mathbf{d}(V^{-1})e\}$. Solving these conditions gives values for the parameters (see Van der Leeuw, 1993). However the computations will become very time consuming when the number of observations is large. This leads to the question whether a more efficient asymptotic form can be found. The asymptotic behaviour depends on the determinant part ($\mathbf{tr}V^{-1}\mathbf{d}V$) and the quadratic form $e^{T}\mathbf{d}(V^{-1})e$. Before we give some theorems regarding these limits we give a lemma about the inverse of the matrix M and a lemma for the asymptotic behaviour of a quadratic form to be used in the sequel.

**Lemma 1**

The inverse of the matrix M can (apart of the lasts rows) be written in block form where the $(i,j)^{th}$ block,

$$M^{-1}_{i,j}=\begin{cases}\underline{M}^{-1} & \text{if } i=j\\ (-\underline{M}^{-1}\underline{N})^{i-j}\underline{M}^{-1} & \text{if } i\geq j\\ 0 & \text{if } i<j\end{cases}$$

$\underline{M}$ refers to the pxp upper left submatrix of M, $\underline{N}$ to the pxp upper part of N. The eigenvalues of $\underline{M}^{-1}\underline{N}$ are less than 1 in absolute value if (and only if) the invertibility condition holds.

**Proof**

Observe that M can be written as $\begin{bmatrix}\underline{M} & & & \\ \underline{N} & \underline{M} & & \\ & \underline{N} & \underline{M} & \\ & & \ddots & \\ & & & \ddots\end{bmatrix}$ where $\underline{M}$ and $\underline{N}$ are as defined above. Direct verification gives the result. For the eigenvalues, see Van der Leeuw (1992). □

**Lemma 2**

$$\lim_{T\to\infty} \mathbf{tr}\{(PM^{-1}N-Q)(\underline{P}^T\underline{P}-\underline{Q}\underline{Q}^T)^{-1}(PM^{-1}N-Q)^T\}/T=0.$$

**Proof**

In the expression $\left(\frac{PM^{-1}N-Q}{\sqrt{T}}\right)(\underline{P}^T\underline{P}-\underline{Q}\underline{Q}^T)^{-1}\left(\frac{PM^{-1}N-Q}{\sqrt{T}}\right)^T$ the (pxp) matrix $(\underline{P}^T\underline{P}-\underline{Q}\underline{Q}^T)$ depends on the AR-parameters and is independent of T. It is non singular (in fact positive definite) because of the invertibility condition. Therefore we have to show that $\lim_{T\to\infty} (PM^{-1}N-Q)/\sqrt{T}=0$.

The $(T\times p)$ matrix $PM^{-1}N-Q$ can be written as

$$(\underline{PM}^{-1}\underline{N}-\underline{Q})\begin{bmatrix} I_p \\ (-\underline{M}^{-1}\underline{N})^1 \\ (-\underline{M}^{-1}\underline{N})^2 \\ \cdot \\ \cdot \\ (-\underline{M}^{-1}\underline{N})^i \\ \cdot \\ \cdot \end{bmatrix},$$ as we will show now. Because N is $\begin{bmatrix}\underline{N}\\ 0\end{bmatrix}$, $M^{-1}N$ is a $(T\times p)$ block column matrix of which the first element is $\underline{M}^{-1}\underline{N}$. The $i^{th}$ block is

$(-\underline{M}^{-1}\underline{N})^{i-1}\underline{M}^{-1}\underline{N}=(-1)^{i-1}(\underline{M}^{-1}\underline{N})^i$, $i=2,3,\ldots$ . Since $P=\begin{bmatrix}\underline{P} & & & \\ \underline{Q} & \underline{P} & & \\ & \underline{Q} & \underline{P} & \\ & & \cdot & \cdot \\ & & & \cdot\ \cdot\end{bmatrix}$, the first

$(p\times p)$ block of $PM^{-1}N-Q$ becomes $\underline{PM}^{-1}\underline{N}-\underline{Q}$, and for subsequent blocks we get:

$\underline{Q}(-1)^{i-2}(\underline{M}^{-1}\underline{N})^{i-1}+\underline{P}(-1)^{i-1}(\underline{M}^{-1}\underline{N})^i$, $i=2,\ 3,\ \ldots$

or

$(\underline{PM}^{-1}\underline{N}-\underline{Q})(-\underline{M}^{-1}\underline{N})^i$, $i=1,2,\ldots$ .

From the invertibility condition we know, that $|\lambda\underline{M}+\underline{N}|$ can only be zero if $|\lambda|<1$. This implies, that the eigenvalues of $\underline{M}^{-1}\underline{N}$ are less than one in absolute value and $\lim_{T\to\infty}(\underline{M}^{-1}\underline{N})^T=0$. But this means, that every block of $PM^{-1}N-Q$ times $1/\sqrt{T}$ tends to zero. □

When T becomes large enough the influence of the 'starting' values of $\bar{\varepsilon}$ and $\bar{v}$ will become less and less important. This implies that the role of the matrices N and Q can be neglected. This has been proven by several authors (*e.g.* Zinde-Walsh, 1992) in a more general context. Here we give a direct proof. It may be more clear if we write V as $W+(N-MP^{-1}Q)(\underline{P}^T\underline{P}-\underline{QQ}^T)^{-1}(N-MP^{-1}Q)^T$, where $W=P^{-1}M(P^{-1}M)^T$. We want to show that asymptotically only W is important in estimating $\beta$ and the ARMA parameters.

However, the determinant of V does not reduce to the determinant of W (which is one). This will be studied in the next section.

**Theorem 1**

Let $V=[N\ M][\bar{P}^T\bar{P}-\bar{Q}\bar{Q}^T]^{-1}[N\ M]^T$ and $W=P^{-1}M(P^{-1}M)^T$.

1. $\text{plim}\ (\hat{\sigma}^2-\tilde{\sigma}^2)=0$, where $\hat{\sigma}^2=\varepsilon^T V^{-1}\varepsilon/T$ and $\tilde{\sigma}^2=\varepsilon^T W^{-1}\varepsilon/T$.
2. If $\lim\limits_{T\to\infty} (X^T W^{-1} X/T)^{-1}$ exists, then $\text{plim}\ (\hat{\beta}-\tilde{\beta})=0$, with $\hat{\beta}$ the Aitken estimator based on V and $\tilde{\beta}$ the estimator based on W.

**Proof**

See appendix. □

For the determinant of the covariance matrix we state the result in theorem 2. Asymptotically the value of the determinant of V tends not to that of W, which is equal to one, but it approaches the value of the determinant of an easy to compute (pxp) matrix.

**Theorem 2**

If the invertibility condition for the MA part holds,

$$\lim_{T\to\infty}|V|=|I_p+(\underline{P}^T\underline{P}-\underline{QQ}^T)^{-1}(\underline{PN}-\underline{MQ})^T(\underline{M}^T\underline{M}-\underline{NN}^T)^{-1}(\underline{PN}-\underline{MQ})|.$$

**Proof**

Because of the structure of M and P we have $|M|=|P|=1$. The determinant of the TxT matrix V is $|I_T+(PM^{-1}N-Q)(\underline{P}^T\underline{P}-\underline{QQ}^T)^{-1}(PM^{-1}N-Q)^T|$, which is equal to the determinant of the pxp matrix $|I_p+(\underline{P}^T\underline{P}-\underline{QQ}^T)^{-1}(PM^{-1}N-Q)^T(PM^{-1}N-Q)|$, as the second part of both expressions have the same non zero eigenvalues. Now, $(PM^{-1}N-Q)^T(PM^{-1}N-Q)=(PN-MQ)^T(M^{-1})^T M^{-1}(PN-MQ)$.

Here PN-MQ is equal to $\begin{bmatrix}\underline{PN}-\underline{MQ}\\ 0\end{bmatrix}$ because $\underline{Q}$ and $\underline{N}$ commute. This implies that $M^{-1}(PN-MQ)$ is equal to the first p columns of $M^{-1}$ times $\underline{PN}-\underline{MQ}$. The elements of this column are $(-\underline{M}^{-1}\underline{N})^{i-1}\underline{M}^{-1}$, i=1,2,... What matters is thus only the upper left block of $(M^{-1})^{T}M^{-1}$, which is equal to $\sum_{i=1}(\underline{M}^{-1})^{T}((\underline{M}^{-1}\underline{N})^{T})^{i-1}(\underline{M}^{-1}\underline{N})^{i-1}(\underline{M}^{-1})$. Let this sum be S. Then

$$\begin{aligned}\underline{M}^{T}S\underline{M}&=\sum_{i=1}((\underline{M}^{-1}\underline{N})^{T})^{i-1}(\underline{M}^{-1}\underline{N})^{i-1}\\ &=I_p+\sum_{i=2}\underline{N}^{T}(\underline{M}^{-1})^{T}((\underline{M}^{-1}\underline{N})^{T})^{i-2}(\underline{M}^{-1}\underline{N})^{i-2}\underline{M}^{-1}\underline{N}\\ &=I_p+\underline{N}^{T}\{\sum_{i=1}(\underline{M}^{-1})^{T}((\underline{M}^{-1}\underline{N})^{T})^{i-1}(\underline{M}^{-1}\underline{N})^{i-1}\underline{M}^{-1}\}\underline{N}\\ &=I_p+\underline{N}^{T}S\underline{N}.\end{aligned}$$

This equation has a solution if $\underline{M}^{T}\otimes\underline{M}^{T}-\underline{N}^{T}\otimes\underline{N}^{T}$ is not singular. Its determinant, say D, is

$$\begin{aligned}D&=|\underline{M}^{T}\otimes\underline{M}^{T}-\underline{N}^{T}\otimes\underline{N}^{T}|\\ &=|I_{p^2}-\underline{N}^{T}\otimes\underline{N}^{T}(\underline{M}^{T}\otimes\underline{M}^{T})^{-1}|\,|\underline{M}^{T}\otimes\underline{M}^{T}|\\ &=|I_{p^2}-(\underline{NM}^{-1})^{T}\otimes(\underline{NM}^{-1})^{T}|.\end{aligned}$$

The eigenvalues of $\underline{NM}^{-1}$ are smaller than 1 in absolute value, implying that the determinant is non singular. Its solution S is equal to $(\underline{M}^{T}\underline{M}-\underline{NN}^{T})^{-1}$ or $(\underline{MM}^{T}-\underline{N}^{T}\underline{N})^{-1}$. These expressions are equivalent as $\underline{M}^{T}\underline{M}+\underline{N}^{T}\underline{N}=\underline{MM}^{T}+\underline{NN}^{T}$: an equality which is shown easily by comparing the products of the commuting matrices $\begin{bmatrix}\underline{N}&\underline{M}\\0&\underline{N}\end{bmatrix}^{T}$ and $\begin{bmatrix}\underline{M}&0\\\underline{N}&\underline{M}\end{bmatrix}$. Then

$S=(\underline{MM}^{T}-\underline{N}^{T}\underline{N})^{-1}\Rightarrow\underline{M}^{T}S\underline{M}=(I_p-\underline{M}^{-1}\underline{N}^{T}\underline{N}(\underline{M}^{-1})^{T})^{-1}$,

$S=(\underline{M}^{T}\underline{M}-\underline{NN}^{T})^{-1}\Rightarrow\underline{N}^{T}S\underline{N}=(\underline{N}^{-1}\underline{M}^{T}\underline{M}(\underline{N}^{-1})^{T}-I_p)^{-1}=((\underline{M}^{-1}\underline{N}^{T}\underline{N}(\underline{M}^{-1})^{T})^{-1}-I_p)^{-1}$, where the (commuting matrices) $\underline{M}^{-1}$ and $\underline{N}^{T}$ are interchanged. Direct verification concludes the proof. □

For the pure AR-case we have $\underline{M}=I_p$ and $\underline{N}=0$. The determinant becomes $|I_p+(\underline{P}^{T}\underline{P}-\underline{QQ}^{T})^{-1}\underline{QQ}^{T}|=|(\underline{P}^{T}\underline{P}-\underline{QQ}^{T})^{-1}[\underline{P}^{T}\underline{P}-\underline{QQ}^{T}+\underline{Q}^{T}\underline{Q}]|=|(\underline{P}^{T}\underline{P}-\underline{QQ}^{T})^{-1}\underline{PP}^{T}|=$

$|(\underline{P}^T\underline{P}-\underline{QQ}^T)^{-1}|$. For the pure MA-case we get a similar result:

$|I_p+\underline{N}^T(\underline{M}^T\underline{M}-\underline{NN}^T)^{-1}\underline{N}|=|\underline{M}^T(\underline{M}^T\underline{M}-\underline{NN}^T)^{-1}\underline{M}|=|(\underline{M}^T\underline{M}-\underline{NN}^T)^{-1}|$.

## 5. Conclusion

Both inverse and determinant of the exact covariance matrix of ARMA distributed errors are rather complicated expressions. However, if the number of observations is large enough to permit an asymptotic approach we can use a simple form for the covariance matrix and its inverse. The value of the determinant used in the concentrated likelihood function reduces to the value of the determinant of an easy to compute matrix of order max(p,q).

## Appendix

### Proof of Theorem 1

1. plim $(\hat{\sigma}^2-\tilde{\sigma}^2)=0$, where $\hat{\sigma}^2=\varepsilon^TV^{-1}\varepsilon/T$ and $\tilde{\sigma}^2=\varepsilon^TW^{-1}\varepsilon/T$.

Straightforward algebra gives:

$$
\begin{aligned}
E(\varepsilon^TW^{-1}\varepsilon-\varepsilon^TV^{-1}\varepsilon)&=E\mathbf{tr}(\varepsilon^TW^{-1}\varepsilon-\varepsilon^TV^{-1}\varepsilon)\\
&=\mathbf{tr}E(W^{-1}\varepsilon\varepsilon^T-V^{-1}\varepsilon\varepsilon^T)\\
&=\sigma^2\mathbf{tr}(W^{-1}V-I_T)\\
&=\sigma^2\mathbf{tr}(W^{-1}(W+(N-MP^{-1}Q)(\underline{P}^T\underline{P}-\underline{QQ}^T)^{-1}(N-MP^{-1}Q)^T)-I_T)\\
&=\sigma^2\mathbf{tr}(P^{-1}M(P^{-1}M)^T)^{-1}(N-MP^{-1}Q)(\underline{P}^T\underline{P}-\underline{QQ}^T)^{-1}(N-MP^{-1}Q)^T\\
&=\sigma^2\mathbf{tr}((P^{-1}M)^T)^{-1}(P^{-1}M)^{-1}(N-MP^{-1}Q)(\underline{P}^T\underline{P}-\underline{QQ}^T)^{-1}(N-MP^{-1}Q)^T\\
&=\sigma^2\mathbf{tr}(P^{-1}M)^{-1}(N-MP^{-1}Q)(\underline{P}^T\underline{P}-\underline{QQ}^T)^{-1}(N-MP^{-1}Q)^T((P^{-1}M)^T)^{-1}\\
&=\sigma^2\mathbf{tr}(PM^{-1}N-Q)(\underline{P}^T\underline{P}-\underline{QQ}^T)^{-1}(PM^{-1}N-Q)^T.
\end{aligned}
$$

In view of lemma 2 this expression divided by T vanishes for $T\to\infty$. □

2. If $\lim_{T\to\infty}(X^TW^{-1}X/T)^{-1}$ exists, then plim $(\hat{\beta}-\tilde{\beta})=0$, with $\hat{\beta}$ the Aitken estimator based on V and $\tilde{\beta}$ the Aitken estimator based on W.

This part needs more manipulation. Write the difference between $\hat{\beta}$ and $\tilde{\beta}$ as a function of the error $\varepsilon$:

$\hat{\beta}=(X^TV^{-1}X)^{-1}XV^{-1}y=(X^TV^{-1}X)^{-1}XV^{-1}(X\beta+\varepsilon)=\beta+(X^TV^{-1}X)^{-1}XV^{-1}\varepsilon$ and

$\tilde{\beta}=\beta+(X^TW^{-1}X)^{-1}XW^{-1}\varepsilon$, which gives for the difference

$\hat{\beta}-\tilde{\beta}=((X^TV^{-1}X)^{-1}XV^{-1}-(X^TW^{-1}X)^{-1}XW^{-1})\varepsilon$ or $Z\varepsilon$.

Observe that $E\|Z\varepsilon\|^2=\mathbf{tr}\ E\ Z\varepsilon\varepsilon^TZ^T=\sigma^2\mathbf{tr}ZVZ^T$. We will show, that $\lim\limits_{T\to\infty} E\|Z\varepsilon\|^2=0$, which implies $\text{plim}(Z\varepsilon)=\text{plim}(\hat{\beta}-\tilde{\beta})=0$, by proving that $\mathbf{tr}Z^TVZ$ has an upper limit similar to the expression in lemma 2. We rewrite $\hat{\beta}-\tilde{\beta}$ replacing V by W and several other symbols. Introduce

$$A=N-P^{-1}MQ, \tag{3}$$

$$\Delta=\underline{P}^T\underline{P}-\underline{QQ}^T, \tag{4}$$

$$H=A^TW^{-1}A+\Delta \tag{5}$$

and

$$G=A^TW^{-1}X(X^TW^{-1}X)^{-1}X^TW^{-1}A-H. \tag{6}$$

Then $V=W+A\Delta^{-1}A^T$, using (3) and (4). This gives for the inverse, using a well known formula for the inverse of the sum of two matrices and (5),

$$\begin{aligned}V^{-1}&=(W+A\Delta^{-1}A^T)^{-1}\\&=W^{-1}-W^{-1}A(A^TW^{-1}A+\Delta)^{-1}A^TW^{-1}\end{aligned}$$

or

$$V^{-1}=W^{-1}-W^{-1}AH^{-1}A^TW^{-1}. \tag{7}$$

Next express $(X^TV^{-1}X)^{-1}$ in W and A:

$$\begin{aligned}(X^TV^{-1}X)^{-1}&=(X^TW^{-1}X-X^TW^{-1}AH^{-1}A^TW^{-1}X)^{-1}\\&=(X^TW^{-1}X)^{-1}-(X^TW^{-1}X)^{-1}X^TW^{-1}A(A^TW^{-1}X(X^TW^{-1}X)^{-1}X^TW^{-1}A-H)^{-1}A^TW^{-1}X(X^TW^{-1}X)^{-1}\end{aligned}$$

or using (6),

$$(X^TV^{-1}X)^{-1}=(X^TW^{-1}X)^{-1}-(X^TW^{-1}X)^{-1}X^TW^{-1}AG^{-1}A^TW^{-1}X(X^TW^{-1}X)^{-1}. \tag{8}$$

Note that G is negative:

$$G=A^TW^{-1}X(X^TW^{-1}X)^{-1}X^TW^{-1}A-H$$
$$=A^TW^{-1}X(X^TW^{-1}X)^{-1}X^TW^{-1}A-A^TW^{-1}A-\Delta$$
$$=A^T(W^{-1}X(X^TW^{-1}X)^{-1}X^TW^{-1}A-W^{-1})A-\Delta$$
$$=-A^T(W^{-1}-W^{-1}X(X^TW^{-1}X)^{-1}X^TW^{-1})A-\Delta.$$

Furthermore we need the expression $W^{-1}VW^{-1}$ rewritten in A and W:

$$W^{-1}VW^{-1}=W^{-1}(W+A\Delta^{-1}A^T)W^{-1}$$

or

$$W^{-1}VW^{-1}=W^{-1}+W^{-1}A\Delta^{-1}A^TW^{-1}. \tag{9}$$

Finally express $ZVZ^T$ in X and W:

$$ZVZ^T=((X^TV^{-1}X)^{-1}XV^{-1}-(X^TW^{-1}X)^{-1}XW^{-1})V((X^TV^{-1}X)^{-1}XV^{-1}-(X^TW^{-1}X)^{-1}XW^{-1})^T$$
$$=(X^TV^{-1}X)^{-1}XV^{-1}VV^{-1}X^T(X^TV^{-1}X)^{-1}-(X^TV^{-1}X)^{-1}XV^{-1}VW^{-1}X^T(X^TW^{-1}X)^{-1}-$$
$$(X^TW^{-1}X)^{-1}XW^{-1}VV^{-1}X^T(X^TV^{-1}X)^{-1}+(X^TW^{-1}X)^{-1}XW^{-1}VW^{-1}X^T(X^TW^{-1}X)^{-1}$$
$$=-(X^TV^{-1}X)^{-1}+(X^TW^{-1}X)^{-1}XW^{-1}VW^{-1}X^T(X^TW^{-1}X)^{-1}.$$

Use (8) and (9) to replace $(X^TV^{-1}X)^{-1}$ and $W^{-1}VW^{-1}$:

$$ZVZ^T=-(X^TW^{-1}X)^{-1}+(X^TW^{-1}X)^{-1}X^TW^{-1}AG^{-1}A^TW^{-1}X(X^TW^{-1}X)^{-1}+$$
$$(X^TW^{-1}X)^{-1}X(W^{-1}+W^{-1}A\Delta^{-1}A^TW^{-1})X^T(X^TW^{-1}X)^{-1}$$
$$=-(X^TW^{-1}X)^{-1}+(X^TW^{-1}X)^{-1}X^TW^{-1}AG^{-1}A^TW^{-1}X(X^TW^{-1}X)^{-1}+$$
$$(X^TW^{-1}X)^{-1}+(X^TW^{-1}X)^{-1}XW^{-1}A\Delta^{-1}A^TW^{-1}X^T(X^TW^{-1}X)^{-1}$$
$$=(X^TW^{-1}X)^{-1}X^TW^{-1}A(G^{-1}+\Delta^{-1})A^TW^{-1}X(X^TW^{-1}X)^{-1}$$
$$<(X^TW^{-1}X)^{-1}X^TW^{-1}A\Delta^{-1}A^TW^{-1}X(X^TW^{-1}X)^{-1},$$

because G is definite negative.

Since we assume that the limit of the inverse of

$X^TW^{-1}X/T=X^T(P^{-1}M(P^{-1}M)^T)^{-1}X/T=(M^{-1}PX/\sqrt{T})^T(M^{-1}PX/\sqrt{T})$ exists, $\lim\limits_{T\to\infty}(M^{-1}PX/\sqrt{T})$

has to exist. Write $\Xi$ for $\left(\dfrac{X^TW^{-1}X}{T}\right)^{-1}\left(\dfrac{(M^{-1}PX)^T}{\sqrt{T}}\right)$.

As $\dfrac{X^TW^{-1}A}{T} = \dfrac{(M^{-1}PX)^T\ M^{-1}P(N-P^{-1}MQ)}{T} = \left(\dfrac{(M^{-1}PX)^T}{\sqrt{T}}\right)\left(\dfrac{PM^{-1}N-Q}{\sqrt{T}}\right)$ we get

$$\lim_{T\to\infty} \mathbf{tr}ZVZ^T < \mathbf{tr}\ \lim_{T\to\infty}\left(\frac{X^TW^{-1}X}{T}\right)^{-1}\left(\frac{X^TW^{-1}A}{T}\right)\Delta^{-1}\left(\frac{A^TW^{-1}X}{T}\right)\left(\frac{X^TW^{-1}X}{T}\right)^{-1}$$

$$< \mathbf{tr}\ \lim_{T\to\infty} \Xi\{(PM^{-1}N-Q)/\sqrt{T}\}\Delta^{-1}\{(PM^{-1}N-Q)^T/\sqrt{T}\}\ \Xi^T$$

$$< \mathbf{tr}\ \lim_{T\to\infty} \Xi\ \lim_{T\to\infty}\{(PM^{-1}N-Q)\Delta^{-1}(PM^{-1}N-Q)^T/T\}\ \lim_{T\to\infty} \Xi^T$$

$=0$, in view of lemma 2. □

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